Asymmetry parameters associated with $F_1$, Table III(a):

$$\begin{aligned} a_{13} &= \tfrac{1}{2}\alpha(p^2 - 1) \\ a_{14} &= \tfrac{1}{2}\gamma(p^2 + 1) \\ a_{23} &= \tfrac{1}{2}\gamma(p^2 - 1) \\ a_{24} &= \tfrac{1}{2}\alpha(p^2 - 1). \end{aligned} \tag{61}$$

Asymmetry parameters associated with $F_2$:

$$\begin{aligned} b_{13} &= \tfrac{1}{2}\alpha(p^2 - 1) \\ b_{14} &= -\tfrac{1}{2}\gamma(p^2 - 1) \\ b_{23} &= \tfrac{1}{2}\gamma(p^2 - 1) \\ b_{24} &= -\tfrac{1}{2}\alpha(p^2 - 1). \end{aligned} \tag{62}$$

Asymmetry parameters associated with $R_2$, Table III(c):

$$g_{13} = g_{23} = g_{14} = g_{24} = 0. \tag{63}$$

Thus, the coupler, the scattering matrix of which had the form (60), has a symmetry (or asymmetry) equivalent to that shown in Fig. 4. For example, if

$$p = \exp\left(-j2\pi \frac{l}{\lambda_g}\right), \tag{64}$$

then the matrix (60) corresponds to that of a coupler which is perfectly symmetrical *except for lengths of guide* $l$ indicated in Fig. 4. The implications as regard dimensional checks or compensating cuts to be made on the component are evident.

# Orthogonality Relationships for Waveguides and Cavities with Inhomogeneous Anisotropic Media*

ALFRED T. VILLENEUVE†

*Summary*—**A modified reciprocity theorem forms the basis of development of orthogonality relationships for modes in waveguides and in cavities containing inhomogeneous, anisotropic media. In the lossless case certain of these may be interpreted in terms of power flow and energy storage. The special case of magnetized gyrotropic media is discussed for longitudinal and transverse magnetization.**

## Introduction

Recently the use of anisotropic materials has been the subject of numerous theoretical and experimental investigations.[1] Such materials are characterized in their macroscopic behavior by tensor permittivities or permeabilities. When these tensors are unsymmetric, the media may be termed "nonreciprocal" since the usual reciprocity theorem[2] does not apply to them. This nonreciprocal behavior finds applications in such devices as circulators, gyrators, load isolators and nonreciprocal phase shifters.[3]

One important special class of nonreciprocal media is that known as gyrotropic media, wherein application of a dc magnetic field causes the permittivity or permeability (hereafter referred to as constitutive parameters) to become an unsymmetric tensor. Two examples are gaseous plasma and ferromagnetic materials, especially low loss, magnetically-saturated ferrites.

Although the usual reciprocity theorem is not valid, a modified reciprocity theorem[4] does apply to anisotropic media. In this theorem, media characterized by transposed tensor constitutive parameters are employed in addition to the original media. In this paper, the modified reciprocity theorem forms a basis for the derivation of orthogonality relationships for modes in waveguides and cavities containing inhomogeneous, anisotropic media.

Let us denote the general form of the constitutive parameters in orthogonal coordinate systems as

$$[\epsilon] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \hat{\epsilon}_{12} & \epsilon_{22} & \epsilon_{23} \\ \hat{\epsilon}_{13} & \hat{\epsilon}_{23} & \epsilon_{33} \end{bmatrix} \quad [\mu] = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \hat{\mu}_{12} & \mu_{22} & \mu_{23} \\ \hat{\mu}_{13} & \hat{\mu}_{23} & \mu_{33} \end{bmatrix}. \tag{1}$$

In this notation the careted symbols, $\hat{\epsilon}_{ij}$ and $\hat{\mu}_{ij}$, are the elements in the $i$th row and $j$th column of the constitutive parameter tensors for media characterized by the transposes of the above tensors. These media shall be referred to as "transposed media." In the case of gyrotropic media this has physical significance, since revers-

* Manuscript received by the PGMTT, March 10, 1959; revised manuscript received April 18, 1959. This work was supported by the Office of Ordnance Research, U. S. Army, Contract No. DA-30-115-ORD-861.

† Hughes Res. and Dev. Lab., Culver City, Calif.

[1] A complete list of references is impractical here and any attempt at making specific references would be difficult. For extensive lists of references the reader is referred to Proc. IRE, vol. 44, pp. 1229–1516; October, 1956.

[2] S. A. Schelkunoff, "Electromagnetic Waves," D. Van Nostrand Co., Inc., New York, N. Y., 1st ed., p. 478; 1943.

[3] C. L. Hogan, "The elements of non-reciprocal microwave devices," Proc. IRE, vol. 44, pp. 1345–1368; October, 1956.

[4] R. F. Harrington and A. T. Villeneuve, "Reciprocity relationships for gyrotropic media," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 308–310; July, 1958.

ing the dc magnetic field transposes these tensors. Field quantities in the transposed media will be denoted by carets and a transposed tensor indicated by a tilde. Our discussion is based on the following form of the modified reciprocity theorem[4]

$$\oint\!\!\oint (\hat{\bar{H}}_a \times \bar{E}_b - \bar{H}_b \times \hat{\bar{E}}_a)\cdot d\bar{S} = \iiint \left[(\bar{J}_a\cdot\bar{E}_b - \bar{K}_a\cdot\bar{H}_b) - (\bar{J}_b\cdot\hat{\bar{E}}_a - \bar{K}_b\cdot\hat{\bar{H}}_a)\right] dv. \quad (2)$$

A second relation, similarly derived, which holds only for the lossless case and which will also be used subsequently is

$$\oint\!\!\oint (\bar{H}_a^* \times \bar{E}_b + \bar{H}_b \times \bar{E}_a^*)\cdot d\bar{S} = \iiint \bar{J}_a^*\cdot\bar{E}_b + \bar{K}_a^*\cdot\bar{H}_b + \bar{J}_b\cdot\bar{E}_a^* + \bar{K}_b\cdot\bar{H}_a^*]dv. \quad (3)$$

$\bar{J}$ and $\bar{K}$ are electric and magnetic source currents.

## General Properties of Modes in Cylindrical Guides

In the following sections orthogonality relationships for modes in waveguides containing inhomogeneous anisotropic media will be discussed. For this discussion it is useful to have some knowledge of general relationships among the mode fields in the original media and in the transposed media. This section is devoted to a study of the field equations for such structures so that these relationships may be investigated.

Let the fields of the various modes be denoted as follows

$$\bar{E}_a = \bar{\mathcal{E}}_a e^{-\gamma_a z}, \qquad \bar{H}_a = \bar{\mathcal{H}}_a e^{-\gamma_a z}. \quad (4)$$

Here it has been assumed that the guide axis is parallel to the $z$ axis and that the structure is uniform; *i.e.*, its material and electrical properties are independent of $z$ (see Fig. 1). These will be referred to as "exponential

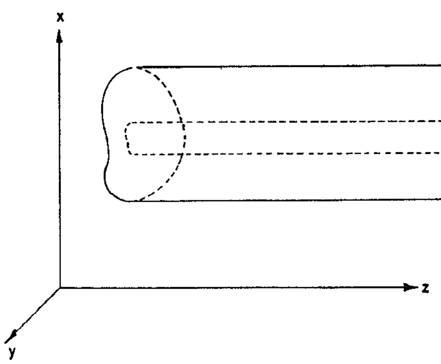


Fig. 1—Section of guide containing inhomogeneous anisotropic media.

modes" hereafter. The $\gamma_a$ represent modes traveling or attenuating in either direction, depending on sign. Before proceeding it is convenient to express the constitutive parameter tensors in a form more suitable to cylindrical structures as follows:

$$[\mu] = \left[\begin{array}{cc|c} \mu_{11} & \mu_{12} & \mu_{13} \\ \hat{\mu}_{12} & \mu_{22} & \mu_{23} \\ \hline \hat{\mu}_{13} & \hat{\mu}_{23} & \mu_{33} \end{array}\right] = \begin{bmatrix} [\mu_{1t}] & [\mu_{2t}] \\ [\tilde{\hat{\mu}}_{2t}] & [\mu_{2z}] \end{bmatrix} \quad (5a)$$

$$[\epsilon] = \left[\begin{array}{cc|c} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \hat{\epsilon}_{12} & \epsilon_{22} & \epsilon_{23} \\ \hline \hat{\epsilon}_{13} & \hat{\epsilon}_{23} & \epsilon_{33} \end{array}\right] = \begin{bmatrix} [\epsilon_{1t}] & [\epsilon_{2t}] \\ [\tilde{\hat{\epsilon}}_{2t}] & [\epsilon_{2z}] \end{bmatrix}. \quad (5b)$$

In these, the elements of the sub-matrices are those of the corresponding partitioned sections. In this notation the field equations become

$$-\nabla_t \times \bar{\mathcal{E}}_{at} = j\omega[\tilde{\hat{\mu}}_{2t}]\bar{\mathcal{H}}_{at} + j\omega[\mu_{2z}]\bar{u}_z\mathcal{H}_{az} \quad (6a)$$

$$\bar{u}_z \times (\gamma_a\bar{\mathcal{E}}_{at} + \nabla_t\mathcal{E}_{az}) = j\omega[\mu_{1t}]\bar{\mathcal{H}}_{at} + j\omega[\mu_{2t}]\bar{u}_z\mathcal{H}_{az}, \quad (6b)$$

$$\nabla_t \times \bar{\mathcal{H}}_{at} = j\omega[\tilde{\hat{\epsilon}}_{2t}]\bar{\mathcal{E}}_{at} + j\omega[\epsilon_{2z}]\bar{u}_z\mathcal{E}_{az}, \quad (6c)$$

$$-\bar{u}_z \times (\gamma_a\bar{\mathcal{H}}_{at} + \nabla_t\mathcal{H}_{az}) = j\omega[\epsilon_{1t}]\bar{\mathcal{E}}_{at} + j\omega[\epsilon_{2t}]\bar{u}_z\mathcal{E}_{az}. \quad (6d)$$

where $\bar{\mathcal{E}}_{at} = \bar{\mathcal{E}}_a - \bar{u}_z\mathcal{E}_{az}$, etc. The differential operators may be considered identical with the usual three dimensional operators, since $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ are independent of $z$. The subscript $t$ on them arises from convention. The boundary conditions at the guide walls are

$$-\hat{n} \times \bar{\mathcal{H}}_a = [y]\bar{\mathcal{E}}_a \quad (7)$$

where $\hat{n}$ is an outward normal and $[y]$ is a tensor admittance. In the transposed media the field equations take the form

$$-\nabla_t \times \hat{\bar{\mathcal{E}}}_{bt} = j\omega[\tilde{\mu}_{2t}]\hat{\bar{\mathcal{H}}}_{bt} + j\omega[\mu_{2z}]\bar{u}_z\hat{\mathcal{H}}_{bz} \quad (8a)$$

$$\bar{u}_z \times (\hat{\gamma}_b\hat{\bar{\mathcal{E}}}_{bt} + \nabla_t\hat{\mathcal{E}}_{bz}) = j\omega[\tilde{\mu}_{1t}]\hat{\bar{\mathcal{H}}}_{bt} + j\omega[\hat{\mu}_{2t}]\bar{u}_z\mathcal{H}_{bz}, \quad (8b)$$

$$\nabla_t \times \hat{\bar{\mathcal{H}}}_{bt} = j\omega[\tilde{\epsilon}_{2t}]\hat{\bar{\mathcal{E}}}_{bt} + j\omega[\epsilon_{2z}]\bar{u}_z\hat{\mathcal{E}}_{bz}, \quad (8c)$$

$$-\bar{u}_z \times (\hat{\gamma}_b\hat{\bar{\mathcal{H}}}_{bt} + \nabla_t\hat{\mathcal{H}}_{bz}) = j\omega[\tilde{\epsilon}_{1t}]\hat{\bar{\mathcal{E}}}_{bt} + j\omega[\epsilon_{2t}]\bar{u}_z\mathcal{E}_{bz}, \quad (8d)$$

subject to the boundary condition

$$-\hat{n} \times \hat{\bar{\mathcal{H}}}_b = [y]\hat{\bar{\mathcal{E}}}_b \quad (9)$$

at the walls.

In ref. 4 it is shown that for every $\gamma_a$ in the original media there exists in the transposed media a propagation constant $\hat{\gamma}_a = -\gamma_a$, with fields which will be denoted $\hat{\bar{\mathcal{E}}}_a$, $\hat{\bar{\mathcal{H}}}_a$. In the general lossy case, there appears to be no simple relation between the fields corresponding to $\gamma_a$ in the original media and those corresponding to $-\gamma_a$ in the transposed media. However, in the lossless case, the following relations hold for the tensor permeability and permittivity[5]

$$[\tilde{\hat{\mu}}_{2t}] = [\tilde{\mu}_{2t}^*], \quad [\tilde{\mu}_{1t}] = [\mu_{1t}^*], \quad [\mu_{2z}] = [\mu_{2z}^*], \quad (10)$$

and, for traveling modes, $-\gamma_a = \gamma_a^*$. Thus, with the relationships

$$\hat{\bar{\mathcal{E}}}_a = \pm\, \bar{\mathcal{E}}_a^* \quad \text{and} \quad \hat{\bar{\mathcal{H}}}_a = \mp\, \bar{\mathcal{H}}_a^*, \quad (11)$$

the field of (8) in the transposed media become the complex conjugates of those of (6) in the original media. Either pair of signs may be selected, and, for convenience, the upper pair will be used in subsequent develop-

[5] This may be seen from energy considerations.

ments. The boundary conditions on the fields are satisfied by this since, for the lossless case,

$$[y] = j[b]. \tag{12}$$

Thus, it is seen that in the lossless case, for every traveling mode in the original media characterized by $\bar{\mathcal{E}}_a$, $\bar{\mathcal{H}}_a$ and $\gamma_a$, there exists in the transposed media a traveling mode characterized by $\hat{\bar{\mathcal{E}}}_a = \bar{\ }_a^*$, $\hat{\bar{\mathcal{H}}}_a = -\bar{\mathcal{H}}_a^*$ and $\hat{\gamma}_a = \gamma_a^* = -\gamma_a$. In the case of evanescent modes in lossless media, $\gamma_a^* = \gamma_a$ and the fields characterized by $\gamma_a$ and $-\gamma_a$ are no longer simply related.

### *Application to Gyrotropic Media*

Because of their useful properties, two special orientations of dc magnetization are commonly employed in nonreciprocal devices containing gyrotropic media. These are the cases of purely longitudinal and purely transverse dc magnetization. The longitudinal case will be considered first. The tensor permeability and permittivity now assume the form

$$[\mu] = \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}; \qquad \epsilon = \begin{bmatrix} \epsilon & -j\eta & 0 \\ j\eta & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}, \tag{13}$$

and it is evident that

$$\begin{aligned} [\mu_{2t}] &= [\epsilon_{2t}] = [0] \\ [\tilde{\mu}_{2t}] &= [\tilde{\epsilon}_{2t}] = [0]. \end{aligned} \tag{14}$$

Under these conditions, the equations in (6) assume the simplified forms

$$-\nabla_t \times \bar{\mathcal{E}}_{at} = j\omega\mu_0 \bar{u}_z \bar{H}_{az} \tag{15a}$$

$$\bar{u}_z \times (\gamma_a \bar{\mathcal{E}}_{at} + \nabla_t \mathcal{E}_{az}) = j\omega[\mu_{1t}]\bar{\mathcal{H}}_{at}, \tag{15b}$$

$$\nabla_t \times \bar{\mathcal{H}}_{at} = j\omega[\epsilon_{2z}]\bar{\mu}_z \mathcal{E}_{az}, \tag{15c}$$

$$-\bar{u}_z \times (\gamma_a \bar{\mathcal{H}}_{at} + \nabla_t \mathcal{H}_{az}) = j\omega[\epsilon_{1t}]\bar{\mathcal{E}}_{at}. \tag{15d}$$

These are the equations of a field characterized by $\bar{\mathcal{E}}_{at}$, $\mathcal{E}_{az}$, $\bar{\mathcal{H}}_{at}$, $\mathcal{H}_{az}$, $\gamma_a$. However, substitution into Maxwell's equations shows that a field characterized by $\bar{\mathcal{E}}_{at}$, $-\mathcal{E}_{az}$, $-\bar{\mathcal{H}}_{at}$, $\mathcal{H}_{az}$ and $-\gamma_a$ also satisfies the equations (15). Examination of boundary conditions shows that they also are satisfied, and the above field is then a possible field in the untransposed media. Thus, when the dc magnetization is purely longitudinal (the anisotropy purely transverse) both $+\gamma_a$ and $-\gamma_a$ are eigenvalues of Maxwell's equations.

### *The Case of Transverse Magnetization*

If the dc magnetization is purely transverse, other properties of the fields become evident. For this case, the permeability and permittivity tensors assume the forms

$$[\mu] = \left[\begin{array}{cc|c} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ \hline -\mu_{13} & -\mu_{23} & \mu_{33} \end{array}\right] = \begin{bmatrix} [\mu_{1t}] & [\mu_{2t}] \\ -[\tilde{\mu}_{2t}] & [\mu_{2z}] \end{bmatrix} \tag{16a}$$

$$[\epsilon] = \left[\begin{array}{cc|c} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \hline -\epsilon_{13} & -\epsilon_{23} & \epsilon_{33} \end{array}\right] = \begin{bmatrix} [\epsilon_{1t}] & [\epsilon_{2t}] \\ -[\tilde{\epsilon}_{2t}] & [\epsilon_{2z}] \end{bmatrix}. \tag{16b}$$

Examination of (6) and (8) shows that, in this case, corresponding to the eigenvalue $\gamma_a$, the fields in the original media and the transposed media are related by

$$\begin{aligned} \hat{\bar{\mathcal{E}}}_{at} &= \bar{\mathcal{E}}_{at}, \quad \hat{\mathcal{E}}_{az} = -\mathcal{E}_{az}, \quad \hat{\bar{\mathcal{H}}}_{at} = -\bar{\mathcal{H}}_{at}, \\ \hat{\mathcal{H}}_{az} &= \mathcal{H}_{az}, \quad \hat{\gamma}_a = -\gamma_a. \end{aligned} \tag{17}$$

This may be thought of as a modified reflectional symmetry relating the original field to the transposed field. This relation, in conjunction with (11) in the lossless case, leads to

$$\begin{aligned} \bar{\mathcal{H}}_{at} &= \bar{\mathcal{H}}_{at}^* \\ \bar{\mathcal{E}}_{at} &= \bar{\mathcal{E}}_{at}^* \\ \mathcal{H}_{az} &= -\mathcal{H}_{az}^* \\ \mathcal{E}_{az} &= -\mathcal{E}_{az}^*. \end{aligned} \tag{18}$$

These equations state that the transverse field components are real and the longitudinal components are imaginary. This means that, *when the dc magnetization is purely transverse, and no loss is present, it is always possible to express the fields in terms of modes whose transverse fields are linearly polarized.*

The results of this section will be applied to simplify certain of the orthogonality relations to be examined.

## Orthogonality Relationships for Cylindrical Guides

In the study of natural modes in closed cylindrical waveguides containing *homogeneous isotropic* media, it is found that the transverse and longitudinal field components satisfy certain orthogonality relationships.[6] Two general types of orthogonality may be considered; those involving vector products of electric and magnetic fields of the various modes, and those involving scalar products of the various mode fields. These orthogonality relationships may be summarized as

$$\iint_s (\bar{\mathcal{E}}_{bt} \times \bar{\mathcal{H}}_{at}) \cdot d\bar{S} = 0 \qquad \gamma_b \neq \pm\gamma_a \tag{19a}$$

$$\iint_s (\bar{\mathcal{E}}_{bt} \times \bar{\mathcal{H}}_{at}^*) \cdot d\bar{S} = 0 \qquad \begin{cases} \gamma_b \neq \pm\gamma_a \\ \gamma_b \neq \pm\gamma_a^* \end{cases} \tag{19b}$$

$$\begin{aligned} &\iint_s \mathcal{E}_{bz}\mathcal{E}_{az}\,dS = \iint_s \bar{\mathcal{E}}_{bt} \cdot \bar{\mathcal{E}}_{at}\,dS = \iint_s \bar{\mathcal{E}}_b \cdot \bar{\mathcal{E}}_a\,dS \\ &= \iint_s \mathcal{H}_{bz}\mathcal{H}_{az}\,dS = \iint_s \bar{\mathcal{H}}_{bt} \cdot \bar{\mathcal{H}}_{at}\,dS = \iint_s \bar{\mathcal{H}}_b \cdot \bar{\mathcal{H}}_a\,dS = 0 \\ &\gamma_b \neq \pm\gamma_a. \end{aligned} \tag{19c}$$

[6] N. Marcuvitz, "Waveguide Handbook," Rad. Lab. Ser. McGraw Hill Book Co., Inc., New York, N. Y., vol. 10, 1st ed., p. 5; 1951.

and finally,

$$\iint_s \mathcal{E}_{bz}\mathcal{E}_{az}^* dS = \iint_s \bar{\mathcal{E}}_{bt}\cdot\bar{\mathcal{E}}_{at}^* dS = \iint_s \bar{\mathcal{E}}_b\cdot\bar{\mathcal{E}}_a^* dS$$
$$= \iint_s \mathcal{H}_{bz}\mathcal{H}_{az}^* dS = \iint_s \bar{\mathcal{H}}_{bt}\cdot\bar{\mathcal{H}}_{at}^* dS$$
$$= \iint_s \bar{\mathcal{H}}_b\cdot\bar{\mathcal{H}}_a^* dS = 0, \qquad \begin{matrix}\gamma_b \neq \pm\gamma_a \\ \gamma_b \neq \pm\gamma_a^*.\end{matrix} \quad (19d)$$

The integrations are performed over guide cross sections. Adler[7] refers to the first two expressions as "power orthogonality" and to the second two groups as "energy orthogonality." The power orthogonality relationships may be derived from the usual reciprocity relationships for isotropic media. The energy relationships appear to result from the fact that in guides containing homogeneous, isotropic media both $\mathcal{E}_z$ and $\mathcal{H}_z$ are solutions of the same scalar Helmholtz equation and satisfy certain boundary conditions at the walls.

*General Orthogonality Relationships for Inhomogeneous Anisotropic Guides*

When the guides contain *inhomogeneous*, isotropic media, the longitudinal field components no longer satisfy the scalar Helmholtz equation and the energy orthogonality relationships no longer hold in general.[8] However, the orthogonality relationship (19a) still holds as a result of reciprocity and reflectional symmetry. Eq. (19b) also holds in the lossless case. If, however, the media are also *anisotropic*, even the power orthogonality relationships must be modified. This is because the usual reciprocity no longer applies and the reflectional symmetry of the arrangement is usually lost. In this case, the modified reciprocity theorem forms the basis of the development. One may begin by considering a source-free region of closed cylindrical guide containing anisotropic media. All material and electrical properties are assumed to be independent of the longitudinal coordinate, which is chosen as $z$. The situation is as shown in Fig. 2. Under these conditions (2) becomes

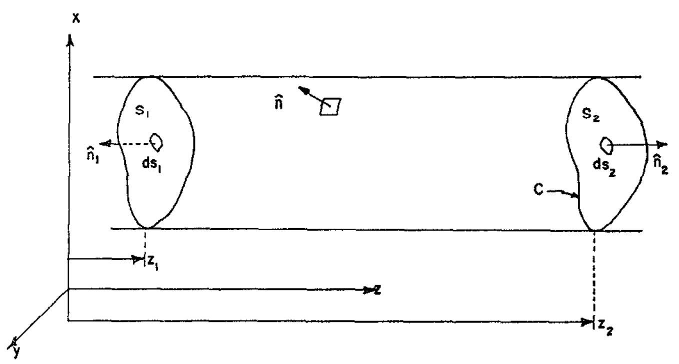


Fig. 2.

[7] R. B. Adler, "Properties of Guided Waves on Inhomogeneous Cylindrical Structures," Res. Lab. of Elec., Mass. Inst. Tech., Cambridge, Tech. Rep. No. 102, pp. 28–29; May 27, 1949.

[8] *Ibid.*, pp. 12 and 29.

$$\oiint_s (\hat{\bar{H}}_b \times \bar{E}_a - \bar{H}_a \times \hat{\bar{E}}_b)\cdot d\bar{S} = 0. \quad (20)$$

The subscripts refer to possible waveguide modes, and the surface of integration is composed of the guide walls and the two cross sections of guide. $d\bar{S}$ is in the direction of the outward normal $\hat{n}$. Let the guide walls be tensor admittance sheets such that

$$-\hat{n} \times \bar{\mathcal{H}}_a = [y]\bar{\mathcal{E}}_a \quad (21a)$$
$$-\hat{n} \times \hat{\bar{\mathcal{H}}}_b = [y]\hat{\bar{\mathcal{E}}}_b. \quad (21b)$$

For the purposes of this paper the tensor $[y]$ will be restricted to the form

$$[y] = \left[\begin{array}{cc|c} y_{11} & y_{12} & 0 \\ y_{12} & y_{22} & 0 \\ \hline 0 & 0 & y_{zz} \end{array}\right] = \begin{bmatrix} [y_{1t}] & [0] \\ [0] & [y_{2z}] \end{bmatrix}, \quad (22)$$

where the division is similar to that of $[\mu]$ and $[\epsilon]$ above. This form is chosen because it leads to symmetrical expressions and is sufficiently general for most purposes.[9] Under these circumstances, for exponential modes, (20) may be reduced to[10]

$$\iint_s (\hat{\bar{\mathcal{H}}}_{bt} \times \bar{\mathcal{E}}_{at} - \bar{\mathcal{H}}_{at} \times \hat{\bar{\mathcal{E}}}_{bt})\cdot d\bar{S} = N_b\delta_{\hat{\gamma}_b,-\gamma_a} \quad (23)$$

where $N_b$ is a normalization constant and $\delta_{ij}$ is the Kronecker delta. The integration is over the guide cross section.

In the lossless case (3), can through an analogous procedure, be put into the following form:

$$\iint_s (\bar{\mathcal{H}}_{bt}^* \times \bar{\mathcal{E}}_{at} + \bar{\mathcal{H}}_{at} \times \bar{\mathcal{E}}_{bt}^*)\cdot d\bar{S} = M_b\delta_{\gamma_b^*,-\gamma_a} \quad (24)$$

where the guide walls are represented by a *lossless* symmetric tensor admittance of the same form as in (22). It may be seen from (11) that for traveling modes in lossless guide $M_b = -N_b$. It should be pointed out, however, that (24) is valid for both traveling and evanescent modes.

Eqs. (23) and (24) are the generalized power orthogonality relationships which hold for guides containing media characterized by tensor permeabilities or permittivities and subject to the appropriate boundary conditions.

*Series Expansion of Fields*

Through use of (23) it is possible to expand an arbitrary transverse field in terms of the transverse fields of exponential modes, assuming that these transverse fields form a complete set. The completeness, however, will not be discussed here. Under this assumption, an

[9] *Ibid.*, pp. 16–17.

[10] This expression and a similar one for purely transverse anisotropy have been obtained by Bresler, Joshi and Marcuvitz from the point of view of the theory of linear operators rather than reciprocity. A. D. Bresler, G. H. Joshi, N. Marcuvitz, "Orthogonality Properties for Modes in Passive and Active Uniform Wave Guides," *J. Appl. Phys.*, vol. 29, pp. 794–798; May, 1958.

arbitrary transverse field may be expressed in series form as

$$\bar{\mathcal{E}}_t = \sum_n A_n \bar{\mathcal{E}}_{nt}; \qquad \bar{\mathcal{H}}_t = \sum_n A_n \bar{\mathcal{H}}_{nt}. \tag{25}$$

On forming the vector products $\hat{\bar{\mathcal{H}}}_{mt} \times \bar{\mathcal{E}}_t$ and $\bar{\mathcal{H}}_t \times \hat{\bar{\mathcal{E}}}_{mt}$, subtracting and integrating over the guide cross section, one gets

$$A_m = \frac{\displaystyle\iint_s (\hat{\bar{\mathcal{H}}}_{mt} \times \bar{\mathcal{E}}_t - \bar{\mathcal{H}}_t \times \hat{\bar{\mathcal{E}}}_{mt}) \cdot d\bar{S}}{N_m}, \tag{26}$$

where it has been assumed that the order of summation and integration may be interchanged. Thus, the coefficients of the expansion are determined.

*Power Flow Relationships*

Eq. (24) may be interpreted in terms of power flow in the guide. This may be seen by considering two distinct modes existing simultaneously in a closed guide. On forming the vector product of $\bar{E}$ and $\bar{H}$ and integrating the longitudinal component over the guide cross section, one arrives at the following result. *The net real power transmitted down a lossless guide is the algebraic sum of the power carried by the individual modes.* The same conclusion may not be drawn about the reactive power, however.

For the general case, (23) and (24) appear to be the only readily available orthogonality expressions. Due to lack of symmetry of $[\mu]$ and $[\epsilon]$, there seems to be no way of reducing them to single cross products as in the isotropic case. In special cases; *e.g.*, in gyrotropic media for special orientations of the dc magnetic field, the tensor properties discussed above provide some additional relationships. These are discussed in the following sections.

*Longitudinal Magnetization of Gyrotropic Media*

It was pointed out above that, for longitudinal dc magnetization of gyrotropic media, both $\gamma_a$ and $-\gamma_a$ are eigenvalues of Maxwell's equations in cylindrical guides and reflectional symmetry exists. Through use of these properties, (23) may be further simplified since one then has the pair of relationships

$$\iint_s (\hat{\bar{\mathcal{H}}}_t \times \bar{\mathcal{E}}_{at} - \bar{\mathcal{H}}_{at} \times \hat{\bar{\mathcal{E}}}_{bt}) \cdot d\bar{S} = N_b \delta_{\hat{\gamma}_b, -\gamma_a} \tag{27a}$$

and

$$\iint_s (\hat{\bar{\mathcal{H}}}_{bt} \times \bar{\mathcal{E}}_{at} + \bar{\mathcal{H}}_{at} \times \hat{\bar{\mathcal{E}}}_{bt}) \cdot d\bar{S} = 0. \tag{27b}$$

Together, these yield the simple orthogonality relationship

$$\iint_s (\hat{\bar{\mathcal{H}}}_{bt} \times \bar{\mathcal{E}}_{at}) \cdot d\bar{S} = -\iint_s (\bar{\mathcal{H}}_{at} \times \hat{\bar{\mathcal{E}}}_{bt}) \cdot d\bar{S} = \tfrac{1}{2} N_b \delta_{\hat{\gamma}_b, \pm\gamma_a}. \tag{28}$$

In the lossless case, (24) leads to the relationship

$$\iint_s (\bar{\mathcal{H}}_{bt}^* \times \bar{\mathcal{E}}_{at}) \cdot d\bar{S} = \iint_s (\bar{\mathcal{H}}_{at} \times \bar{\mathcal{E}}_{bt}^*) \cdot d\bar{S} = \tfrac{1}{2} M_b \delta_{\gamma_b^*, \pm\gamma_a}. \tag{29}$$

This last equation may be interpreted in terms of power flow as follows. *The net complex power transmitted down a lossless guide whose anisotropy is purely transverse is the algebraic sum of the complex powers of the individual modes.* This statement is more specific than could be made for the general case, since it now includes the *reactive* power. The general case included only the real power.

*Transverse Magnetization of Gyrotropic Media*

In view of the relationship between fields in the original and in the transposed media for transverse magnetization, the power orthogonality relationship (23) may be rewritten as

$$\iint_s [\bar{\mathcal{H}}_{bt} \times \bar{\mathcal{E}}_{at} + \bar{\mathcal{H}}_{at} \times \bar{\mathcal{E}}_{bt}] \cdot d\bar{S} = -N_b \delta_{\hat{\gamma}_b, -\gamma_a}. \tag{30}$$

This special case involves the fields in only the original media.

*Energy Type Relationships*

Through use of the field equations (6) and (8) and the application of various vector identities, it is possible to rewrite the preceding power orthogonality relationships in various other forms. Under certain conditions in the lossless case, these may be interpreted in terms of stored energy. The details are long and involved and again, only results are presented here for the special case of purely transverse anisotropy.[11] In this special case, the form of $[\mu]$ and $[\epsilon]$ make it possible to derive the relationship

$$\iint_s (\hat{\bar{\mathcal{E}}}_b \cdot [\epsilon] \cdot \bar{\mathcal{E}}_a + \hat{\bar{\mathcal{H}}}_b \cdot [\mu] \bar{\mathcal{H}}_a) dS + \frac{1}{j\omega} \oint_C \hat{\bar{\mathcal{E}}}_b \cdot [y] \bar{\mathcal{E}}_a dl = 0, \qquad \text{all } \hat{\gamma}_b, \gamma_a. \tag{31}$$

Here $C$ is the perimeter of the guide cross section. In the lossless case, one gets the corresponding relationship

$$\iint_s \bar{\mathcal{E}}_b^* \cdot [\epsilon] \bar{\mathcal{E}}_a dS + \frac{1}{\omega} \oint_C \bar{\mathcal{E}}_b^* \cdot [b] \bar{\mathcal{E}}_a dl = \iint_s \bar{\mathcal{H}}_b^* \cdot [\mu] \bar{\mathcal{H}}_a dS, \qquad \text{all } \gamma_b^*, \gamma_a. \tag{32}$$

This equation may be interpreted in terms of mutual time-average electric and magnetic energies of two

[11] For details see in A. T. Villeneuve, "A study of Reciprocity Relationships For Gyrotropic Media," Res. Inst., Syracuse University, Syracuse, N. Y., Final Rept. No. EE509-589F; September, 1958.

modes, if one interprets the integral over $C$ as the electric energy stored in the guide walls per unit length. With this interpretation, (32) states that *the mutual time average stored electric and magnetic energies of two modes are equal.*

From the foregoing, it is evident that when cylindrical guides contain *inhomogeneous, anisotropic* media, most of the usual orthogonality relationships are lost. Only a modified power orthogonality relationship remains in general. The energy orthogonality relationships are destroyed. However, when the anisotropy is purely transverse, the orthogonality relationships are similar to those which exist for the inhomogeneous, *isotropic* case.

### *Orthogonality Relationships for Closed Cavities*

Next, orthogonality relationships for closed cavities containing inhomogeneous anisotropic media will be investigated. Consider such a closed cavity with perfectly conducting walls as in Fig. 3.

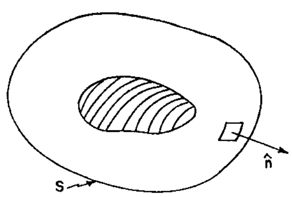


Fig. 3—Cavity with inhomogeneous anistropic media.

The usual orthogonality relationships among the modes[12] do not hold, except, possibly, in special cases. However, modified orthogonality relationships do exist. It has been demonstrated from the modified reciprocity theorem that the natural resonant frequencies of such a cavity are identical in both the transposed and the original media.[13] This forms the basis of the modified orthogonality conditions as follows. In the original cavity there exists a set of natural modes characterized by $\overline{E}_i$, $\overline{H}_i$ and $\omega_i$, where $\omega_i=\omega_i'-j\omega_i''$ in general. In the cavity with media transposed (transposed cavity), one has modes characterized by $\hat{\overline{E}}_i$, $\hat{\overline{H}}_i$ and $\omega_i$. Note that the $\omega_i$ are the same in both cases. The fields in the original cavity satisfy the equations

$$-\nabla\times\overline{E}_i = j\omega_i[\mu]\overline{H}_i \tag{33a}$$

$$\nabla\times\overline{H}_i = j\omega_i[\epsilon]\overline{E}_i \tag{33b}$$

$$\nabla\times[\mu]^{-1}\nabla\times\overline{E}_i = \omega_i^2[\epsilon]\overline{E}_i \tag{34a}$$

$$\nabla\times[\epsilon]^{-1}\nabla\times\overline{H}_i = \omega_i^2[\mu]\overline{H}_i \tag{34b}$$

subject to the boundary condition that the tangential component of $\overline{E}_i$ vanish at the walls. Similar equations are satisfied by $\hat{\overline{E}}_i$ and $\hat{\overline{H}}_i$ in the transposed case except that the tensor constitutive parameters are transposed.

In order to study the orthogonality relationships, one may proceed as follows. First, form the volume integral over the cavity

$$\iiint (\hat{\overline{E}}_m\cdot\nabla\times[\mu]^{-1}\nabla\times\overline{E}_n - \overline{E}_n\cdot\nabla\times[\tilde{\mu}]^{-1}\nabla\times\hat{\overline{E}}_m)dv. \tag{35}$$

By the use of vector identities, the divergence theorem, and (33) and (34), for both the original and transposed media and boundary conditions on $\overline{E}$, one arrives at the orthogonality relationship

$$\iiint \hat{\overline{E}}_m\cdot[\epsilon]\overline{E}_n dv = \delta_{mn}, \tag{36}$$

where the fields have been normalized and $\delta_{mn}$ is the Kronecker delta. Through a similar procedure, one may also arrive at the result

$$-\iiint \hat{\overline{H}}_n\cdot[\mu]\overline{H}_n dv = \delta_{mn}, \tag{37}$$

the $\overline{H}_n$ being automatically normalized when the $\overline{E}_n$ are normalized. In the lossless case, these reduce to

$$\iiint \overline{E}_m^*\cdot[\epsilon]\overline{E}_n dv = \iiint \overline{H}_m^*\cdot[\mu]\overline{H}_n dv = \delta_{mn}. \tag{38}$$

Eq. (38) shows that *in closed cavities containing lossless anisotropic media, the total electric or magnetic energy is the sum of the energies of the individual modes with no coupling terms between modes.* However, when loss is present, the above expressions do not necessarily hold and some mutual energy terms may be present.

It may be seen from the foregoing that orthogonality relationships for perfectly conducting cavities containing inhomogeneous anisotropic media are quite similar to those for isotropic media except that fields in the transposed cavity must be used in addition to those in the original cavity. Only in the lossless case can energy interpretations be given to these relationships, for in this case the fields in the original and in the transposed media are simply related.

ACKNOWLEDGMENT

The author wishes to acknowledge the suggestions and encouragement of Dr. R. F. Harrington, under whose direction this study was carried out.

[12] G. Toraldo di Francia, "Electromagnetic Waves," Interscience Publishers, Inc., New York, N. Y., 1st ed., p. 303; 1955.

[13] R. F. Harrington and A. T. Villeneuve, *op. cit.*